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REMARKS ON NAVIER-STOKES EQUATIONS WITH MEASURES AS DATA

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Abstract—We show the existence of a solution to the Navier-Stokes equation taking the vorticity ω as the unknown: $\omega_t + A\omega + B\omega = \mu$, $\omega(0) = \omega_0$. Here, ω_0 and μ are bounded Radon measures. This study is motivated by a numerical approximation which will be given in a forthcoming work [1].

1. INTRODUCTION

Usually, in the Navier-Stokes equations for an incompressible fluid, the unknowns are the velocity V and the pressure p (see [2]):

$$(P_1)\begin{cases} \frac{\partial V}{\partial t} - \nu\Delta V + (V\cdot\nabla)V + \operatorname{grad} p = f \text{ in } Q = (0,T)\times\Omega, \\ \operatorname{div} V = 0 \text{ in } Q. \end{cases}$$

Assume $\Omega \subset \mathbb{R}^2$, physicists also consider as unknowns the vorticity $\omega = (\operatorname{curl} V)\vec{k}$ and the stream function ψ satisfying $V = \operatorname{curl}(\psi\vec{k})$, where $\vec{k}$ is the unit vector orthogonal to the plane $\mathbb{R}^2$. If we take the "curl" of the first equation and take the projection on the k-axis, we get from (P_1):

$$(P_2)\begin{cases} \frac{\partial \omega}{\partial t} - \nu\Delta\omega + (V\cdot\nabla)\omega = \mu \text{ in } Q, \\ -\Delta\psi = \omega \text{ in } Q. \end{cases}$$

We supplement (P_2) with the additional conditions that $\omega = \psi = 0$ on $(0,T)\times\partial\Omega$ and $\omega(0,\cdot) = \omega_0$; ω_0 as well as μ are bounded Radon measures on Ω. If $\Omega = \mathbb{R}^2$, this problem was already considered by G.H. Cottet [3,4] when $\mu = 0$. So, we will assume that Ω is a bounded domain. Although the boundary condition on ω is not the usual one it is here a relevant one as our study is above all motivated by the numerical approximation of (P_2) in a large domain Ω.

2. NOTATIONS AND MAIN RESULTS

We denote by $M(Q)$ (resp., $M(\Omega)$) the set of bounded Radon measures on Q (resp., on Ω), and we set for $m \geq 1$ and $p \in]1,+\infty]$;

$$Z^{m,p}(\Omega) = \bigcap_{s\in[1,p[} W^{m,s}(\Omega), \qquad Z_0^{m,p}(\Omega) = Z^{m,p}(\Omega) \cap W_0^{1,1}(\Omega).$$

Now, we define the B-functions on $Z_0^{1,4/3}(\Omega)$ by first introducing the map

$$L : Z_0^{1,4/3}(\Omega) \mapsto Z_0^{2,4}(\Omega) \text{ with } L\omega = \psi \text{ if and only if } -\Delta\psi = \omega \text{ in } \Omega, \psi = 0 \text{ on } \partial\Omega.$$

We set $B(\omega,\varphi) = \frac{\partial}{\partial x_2}L\omega\frac{\partial\varphi}{\partial x_1} - \frac{\partial}{\partial x_1}L\omega\frac{\partial}{\partial x_2}$ for (ω,φ) in $Z_0^{1,4/3}(\Omega)\times Z_0^{1,4/3}(\Omega)$ and $B\omega = B(\omega,\omega)$.

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As a property of B, we have the following.

LEMMA 1. *Let Φ be a bounded lipschitz function from $\mathbb{R}$ into $\mathbb{R}$, then for all $\omega \in Z_0^{1,4/3}(\Omega)$, we have*

(i) $B\omega \in L^p(\Omega)$, $1 \le p < \frac{4}{3}$;
(ii) $\int_\Omega B\omega(x)\Phi(\omega(x))\,dx = 0$;
(iii) $\forall \varphi \in W_0^{1,+\infty}(\Omega)$, $\int_\Omega B\omega\varphi\,dx = -\int_\Omega \omega B(\omega,\varphi)\,dx$.

PROOF. For $\omega \in Z_0^{1,4/3}(\Omega)$, $\operatorname{grad}(L\omega) \in L^\infty(\Omega)^2$; so we have (i). We write

$$\int_\Omega B\omega\Phi(\omega)\,dx = \int_\Omega \frac{\partial}{\partial x_2} L\omega \frac{\partial}{\partial x_1}\left(\int_0^\omega \Phi(\sigma)\,d\sigma\right) dx - \int_\Omega \frac{\partial}{\partial x_1} L\omega \frac{\partial}{\partial x_2}\left(\int_0^\omega \Phi(\sigma)\,d\sigma\right) dx.$$

By integration by parts, we get (ii), and (iii) is also derived by an integration by parts.

The problem (P_2) is equivalent to solving the problem $\omega_t + A\omega + B\omega = \mu$ on Q, $\omega(t) \in Z_0^{1,\frac{4}{3}}(\Omega)$ for a.e., $t \in [0,T]$ where $A\omega = -\nu\Delta\omega$, $\nu > 0$.

THEOREM 1. *(Existence of a weak solution) Let $\mu \in M(Q)$ and $\omega_0 \in M(\Omega)$, then there exists $\omega \in \bigcap_{q\in[1,\frac{4}{3}[} L^q(0,T;W_0^{1,q}(\Omega))$ satisfying*

$$-\int_Q \omega\varphi_t\,dx\,dt - \int_\Omega \varphi(0,x)\,d\omega_0 + \nu\int_Q \nabla\omega\nabla\varphi\,dx\,dt - \int_Q \omega B(\omega,\varphi)\,dx\,dt = \int_Q \varphi(t,x)\,d\mu,$$

for all $\varphi \in C_0^\infty(\mathbb{R}^3)$, which is zero in a neighborhood of $(0,T)\times\partial\Omega$ and $\{T\}\times\Omega$.

REMARK. We prove below that $\omega B(\omega,\varphi)$ is in $L^1(Q)$.

3. PROOF OF THEOREM 1

We follow the same scheme as in [5]. First of all, we consider μ^ε in $C_0^\infty(Q)$ (resp., ω_0^ε in $C_0^\infty(\Omega)$) with μ^ε (resp., ω_0^ε) remaining in a bounded set of $L^1(Q)$ (resp., in $L^1(\Omega)$), $(\varepsilon > 0)$. Using the result in R. Temam's book [2], we derive the existence of a function $\omega^\varepsilon \in L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$ satisfying: (P_ε) : $\omega_t^\varepsilon + A\omega^\varepsilon + B\omega^\varepsilon = \mu^\varepsilon$ in $\mathcal{D}'(Q)$, $\omega^\varepsilon(0) = \omega_0^\varepsilon$.

From Lemma 1 and integration by parts, we derive the following lemma.

LEMMA 2. *Let Φ be a bounded lipschitz function from $\mathbb{R}$ into $\mathbb{R}$, then for all $t \in [0,T]$:*

$$\int_\Omega dx \int_0^{\omega^\varepsilon(t,x)} \Phi(\sigma)\,d\sigma - \int_\Omega dx\int_0^{\omega_0^\varepsilon(x)} \Phi(\sigma)\,d\sigma + \nu\int_0^t\int_\Omega \nabla\omega^\varepsilon\nabla\Phi(\omega^\varepsilon)\,dx\,d\sigma = \int_0^t\int_\Omega \Phi(\omega^\varepsilon)\mu^\varepsilon\,dx\,d\sigma.$$

Choosing various functions Φ, as in [5], in Lemma 2 and applying Lemma 1 of [5], we deduce the following lemma.

LEMMA 3. *ω^ε remains in a bounded set of $L^q(0,T;W_0^{1,q}(\Omega))$ for all $q \in [1,\frac{4}{3}[$ and also in $L^\infty(0,T;L^1(\Omega))$ as $\varepsilon \longrightarrow 0$.*

LEMMA 4. *There exists $r > 1$ such that $\omega^\varepsilon \operatorname{grad}(L\omega^\varepsilon)$ remains in a bounded set of $L^r(\Omega)^2$.*

PROOF OF LEMMA 4. For a fixed $q \in]1,\frac{4}{3}[$, we set $q^* = 2q/2-q$. Consider $\eta > 0$ such that $q^*\eta < (q-1)(q^*-1)(1+\eta)$. By interpolation and Lemma 3, we have

$$|\omega^\varepsilon(t)|_{L^{1+\eta}(\Omega)} \le |\omega^\varepsilon(t)|_{L^1(\Omega)}^\theta |\omega^\varepsilon(t)|_{L^{q^*}(\Omega)}^{1-\theta} \le c|\omega^\varepsilon(t)|_{L^{q^*}(\Omega)}^{1-\theta}, \qquad \forall\, t > 0.$$

Note that $1-\theta = q^*\eta/(1+\eta)(q^*-1)$. From the definition of $L\omega^\varepsilon$, Calderon-Zygmund's theorem and the preceding estimate, we get:

$$|\omega^\varepsilon(t)\mathrm{grad}(L\omega^\varepsilon)(t)|_{L^{1+\eta}(\Omega)} \le |\mathrm{grad}(L\omega^\varepsilon)(t)|_{L^\infty(\Omega)}|\omega^\varepsilon(t)|_{L^{1+\eta}(\Omega)}$$
$$\le c|\omega^\varepsilon(t)|_{L^{q^*}(\Omega)}|\omega^\varepsilon(t)|^{1-\theta}_{L^{q^*}(\Omega)} = c|\omega^\varepsilon(t)|^{2-\theta}_{L^{q^*}(\Omega)}.$$

Since $q/(2-\theta) > 1$, we have for $r = \inf(q/2-\theta,\ 1+\eta)$,

$$|\omega^\varepsilon \mathrm{grad}(L\omega^\varepsilon)|_{L^r(Q)} \le c|\omega^\varepsilon|^{2-\theta}_{L^q(0,T;W_0^{1,q}(\Omega))} \le \text{ constant.}$$

COROLLARY OF LEMMAS 3 AND 4.

(i) *There exits $s>1$ such that: ω_t^ε remains in a bounded set of $L^1(0,T;H^{-s}(\Omega))$.*
(ii) *For all φ in $C^\infty(\overline{Q})$, $\omega^\varepsilon B(\omega^\varepsilon,\varphi)$ remains in a bounded set of $L^r(Q)$ (r as in Lemma 4).*

The proof is straightforward.

PROOF OF THEOREM 1. We can assume that there exists a function ω such that

(i) ω^ε converges weakly to ω in $L^q(0,T;W_0^{1,q}(\Omega))$ for all $q \in [1, \frac{4}{3}[$.
(ii) ω^ε converges to ω almost everywhere in Q (Use the compactness' result of [2]).
(iii) $\mathrm{grad}(L\omega^\varepsilon)$ converges to $\mathrm{grad}(L\omega)$ almost everywhere in Q (use the definition of L, Calderon-Zygmund theorem and the preceding point (ii)).
(iv) For all φ in $C^\infty(\overline{Q})$, $\omega^\varepsilon B(\omega^\varepsilon,\varphi)$ converges to $\omega B(\omega,\varphi)$ in $L^1(Q)$-strong.

Multiplying by a suitable function φ (as in Theorem 1) and passing to the limit in (P_ε), we prove Theorem 1.

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